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MULTIPLICATIVE REPRESENTATIONS OF INTEGERS

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ABSTRACT

Let $h \geqq 2$, and let $\mathscr{B} = (B_1, \ldots, B_h)$, where $B_i \subseteq \mathbf{N} = \{1, 2, 3, \ldots\}$ for $i = 1, \ldots, h$. Denote by $g_{\mathscr{B}}(n)$ the number of representations of n in the form $n = b_1 \cdots b_h$, where $b_i \in B_i$. If $g_{\mathscr{B}}(n) > 0$ for all $n > n_0$, then $\mathscr{B}$ is an *asymptotic multiplicative system of order h*. The set B is an *asymptotic multiplicative basis of order h* if $n = b_1 \cdots b_h$ is solvable with $b_i \in B$ for all $n > n_0$. Denote by $g(n)$ the number of such representations of n. Let $M(h)$ be the set of all pairs (s, t), where $s = \liminf_{n \to \infty} g_{\mathscr{B}}(n)$ and $t = \limsup_{n \to \infty} g_{\mathscr{B}}(n)$ for some multiplicative system $\mathscr{B}$ of order h. It is proved that

$$M(h) = \{(1, t) \mid t \in \mathbf{N}\} \cup \{(s, \infty) \mid s = 1, \ldots, h\}.$$

In particular, it follows that $s \geqq 2$ implies $t = \infty$. A corollary is a theorem of Erdös that if B is a multiplicative basis of order $h \geqq 2$, then $\limsup_{n \to \infty} g(n) = \infty$. Similar results are obtained for asymptotic union bases of finite subsets of $\mathbf{N}$ and for asymptotic least common multiple bases of integers.

1. Introduction

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. For $h \geqq 2$, let $\mathscr{B} = (B_1, B_2, \ldots, B_h)$ be an h-tuple of subsets $B_i \subseteq \mathbf{N}$ for $i = 1, \ldots, h$. Denote by $g_{\mathscr{B}}(n)$ the number of representations of n in the form

$$n = b_1 b_2 \cdots b_h \tag{1}$$

where $b_i \in B_i$ for $i = 1, \ldots, h$. If $g_{\mathscr{B}}(n) > 0$ for all sufficiently large n, then $\mathscr{B}$ is an *asymptotic multiplicative system of order h*.

If B is a set of natural numbers such that the h-tuple $\mathscr{B} = (B, B, \ldots, B)$ is an asymptotic multiplicative system of order h, then the set B is called an *asymptotic multiplicative basis of order h*. If every $n \geqq 1$ can be represented in the

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form (1) with $b_i \in B$ for $i = 1, \ldots, h$, then B is a *multiplicative basis of order h*. Erdős [2] and Nešetřil and Rüdl [6] proved that if B is a multiplicative basis of order h, then $\limsup_{n\to\infty} g_{\mathscr{B}}(n) = \infty$. This is the multiplicative analog of an old problem in additive number theory. For $B \subseteq \mathbf{N} \cup \{0\}$, let $f(n)$ denote the number of representations of n in the form $n = b_1 + b_2$, where $b_1, b_2 \in B$. Erdős and Turán [3] conjectured that $f(n) > 0$ for $n > n_0$ implies $\limsup_{n\to\infty} f(n) = \infty$. The conjecture is still unsolved.

In this paper I generalize Erdős's theorem on multiplicative bases to the case of asymptotic multiplicative systems. It is possible to construct h-tuples $\mathscr{B} = (B_1, \ldots, B_h)$ such that

$$0 < \liminf_{n\to\infty} g_{\mathscr{B}}(n) \leqq \limsup_{n\to\infty} g_{\mathscr{B}}(n) < \infty.$$

Here are two examples. Partition the set $\mathbf{P}$ of primes into h pairwise disjoint sets $P_1, \ldots, P_h$, and let B_i be the set of all $n \in \mathbf{N}$ such that if $p \in \mathbf{P}$ and $p \mid n$, then $p \in P_i$. Every natural number n has a unique representation in the form (1), and so $g_{\mathscr{B}}(n) = 1$ for all $n \geqq 1$.

A second example: Let $\mathscr{B} = (B_1, \ldots, B_h)$, where $B_1 = \mathbf{N}$, $B_2 = \{1, 2\}$, and $B_i = \{1\}$ for $i = 3, \ldots, h$. Then $g_{\mathscr{B}}(n) = 1$ if n is odd and $g_{\mathscr{B}}(n) = 2$ if n is even.

These examples show that $\liminf_{n\to\infty} g_{\mathscr{B}}(n) \geqq 1$ does *not* imply that $\limsup_{n\to\infty} g_{\mathscr{B}}(n) = \infty$. Let $M(h)$ consist of all pairs (s, t) such that $s = \liminf_{n\to\infty} g_{\mathscr{B}}(n)$ and $t = \limsup_{n\to\infty} g_{\mathscr{B}}(n)$ for some asymptotic multiplicative system $\mathscr{B}$ of order h. The principal result of this paper is the explicit description of $M(h)$. In particular, it follows from this characterization of $M(h)$ that $\liminf_{n\to\infty} g_{\mathscr{B}}(n) \geqq 2$ implies that $\limsup_{n\to\infty} g_{\mathscr{B}}(n) = \infty$. Erdős's theorem on multiplicative bases is an immediate corollary of this result.

In this paper I also consider asymptotic union bases. Let $\mathscr{F}(\mathbf{N})$ denote the set of all finite subsets of $\mathbf{N}$, and let $\mathscr{A}_i \subseteq \mathscr{F}(\mathbf{N})$ for $i = 1, \ldots, h$. If for all but at most finitely many $S \in \mathscr{F}(\mathbf{N})$ there exist sets $A_i \in \mathscr{A}_i$ such that

$$(2) \qquad S = A_1 \cup \cdots \cup A_h$$

then $\mathscr{A}^* = (\mathscr{A}_1, \ldots, \mathscr{A}_h)$ is an asymptotic union system of order h. If $\mathscr{A}_i = \mathscr{A} \subseteq \mathscr{F}(\mathbf{N})$ for $i = 1, \ldots, h$, then $\mathscr{A}$ is an *asymptotic union basis of order h*. Union bases have been studied by Deza and Erdős [1], Grekos [4], and Nathanson [5].

Partition $\mathbf{N}$ into h pairwise disjoint sets N_i, let $\mathscr{A}_i$ denote the set of all finite subsets of N_i, and let $\mathscr{A}^* = (\mathscr{A}_1, \ldots, \mathscr{A}_h)$. Then every $S \in \mathscr{F}(\mathbf{N})$ has a unique representation in the form (2).

Let $\mathscr{A}^*$ be an asymptotic union system of order h, and let $r(S)$ denote the number of representations of S in the form (2). Let $\mathcal{F}(\mathbf{N}) = \{S_n\}_{n=1}^{\infty}$. I shall prove that if $\liminf_{n\to\infty} r(S_n) \geqq 2$, then $\limsup_{n\to\infty} r(S_n) = \infty$.

The problem of representing a finite set as the union of h subsets is identical to the problem of representing a square-free integer q as the least common multiple of h divisors of q. This observation leads to the study of least common multiple bases for the integers. Let $\mathcal{B} = (B_1, \ldots, B_h)$, where $B_i \subseteq \mathbf{N}$ for $i = 1, \ldots, h$. If every sufficiently large integer n can be represented in the form

$$n = [b_1, \ldots, b_h] \tag{3}$$

where $b_i \in B_i$ for $i = 1, \ldots, h$ and $[b_1, \ldots, b_h]$ denotes the least common multiple (LCM) of the integers b_i, then $\mathcal{B}$ is an *asymptotic LCM system of order h*. If $\mathcal{B}$ is an asymptotic LCM system such that $B_i = B \subseteq \mathbf{N}$ for all $i = 1, \ldots, h$, then the set B is an *asymptotic LCM basis of order h*. Nathanson [5] introduced LCM bases and nonbases.

Let $\mathcal{B}$ be an asymptotic LCM system of order h, and let $g'_{\mathcal{B}}(n)$ denote the number of representations of n in the form (3). I shall prove that $\liminf_{n\to\infty} g'_{\mathcal{B}}(n) \geqq 2$ implies $\limsup_{n\to\infty} g'_{\mathcal{B}}(n) = \infty$.

Notation. Let $\mathbf{N}$ denote the set of natural numbers and $\mathbf{P}$ the prime numbers. For $n \in \mathbf{N}$ and $p \in \mathbf{P}$, write $p^k \,\|\, n$ if p^k is the highest power of p that divides n.

Let X be a countably infinite set. If $S \subseteq X$, let $|S|$ denote the cardinality of S. For $d \in \mathbf{N} \cup \{0\}$, let $[X]^d$ denote the collection of all $S \subseteq X$ with $|S| = d$. Let $[X]^{<\omega}$ denote the set of all finite subsets of X. Thus $\mathcal{F}(\mathbf{N}) = [\mathbf{N}]^{<\omega}$.

2. Ramsey's theorem and union bases

The main tool used in this paper is the following lemma, which is an application of Ramsey's theorem in combinatorial analysis. The lemma is a refinement of a result of Nešetřil and Rüdl [6].

LEMMA. *Let $h \geqq 2$, let X be a countably infinite set, and let $\mathcal{A}^* = (\mathcal{A}_1, \ldots, \mathcal{A}_h)$, where $\mathcal{A}_i \subseteq [X]^{<\omega}$ for $i = 1, \ldots, h$. Suppose that the h-tuple $\mathcal{A}^*$ satisfies the following condition:*

There exist infinitely many $d \in \mathbf{N}$ such that for all but at most finitely many sets $S \in [X]^d$ the set theoretic equation

$$S = A_1 \cup \cdots \cup A_h$$

has at least two solutions with

$$A_i \in \mathcal{A}_i \quad \text{for } i = 1, \ldots, h.$$

Then for every $n \in \mathbf{N}$ *there is a set* $T \in [X]^{<\omega}$ *with the property that there exist sets* $A_{i,k} \in \mathcal{A}_i$ *for* $k = 1, \ldots, n$ *such that*

$$T = A_{1,k} \cup \cdots \cup A_{h,k}$$

for $k = 1, \ldots, n$, *and*

$$A_{i,k} \cap A_{j,k} = \emptyset$$

for $1 \leqq i < j \leqq h$.

Proof. Let $n \in \mathbf{N}$. Choose $d \geqq n$ such that the condition of the Lemma holds for sets $S \in [X]^d$.

For $r = 0, 1, \ldots, d$, let

$$[X]^r = \bigcup_{i=0}^{h} C_i^r$$

where for $S \in [X]^r$

$$S \in \begin{cases} C_0^r & \text{if } S \notin \bigcup_{i=1}^{h} \mathcal{A}_i, \\ C_i^r & \text{if } S \in \mathcal{A}_i. \end{cases}$$

Ramsay's theorem implies that there is an infinite subset $Y \subseteq X$ such that $[Y]^r$ is homogeneous with respect to the sets $\{C_i^r\}_{i=0}^{h}$ for $r = 0, 1, \ldots, d$. This means that for each r there is an i such that $[Y]^r \subseteq C_i^r$.

Since d satisfies the condition of the Lemma, there is a set $S \in [Y]^d$ such that there exist sets $A_i, A'_i \in \mathcal{A}_i$ for $i = 1, \ldots, h$, where

$$S = A_1 \cup \cdots \cup A_h = A'_1 \cup \cdots \cup A'_h$$

and $A_i \neq A'_i$ for some i. Let $d_i = |A_i|$ and $d'_i = |A'_i|$ for $i = 1, \ldots, h$. Then $0 \leqq d_i, d'_i \leqq d$ for all i, and $d \leqq \Sigma_{i=1}^{h} d_i$ and $d \leqq \Sigma_{i=1}^{h} d'_i$. (Note that the Lemma does not assume that the sets A_i or A'_i are pairwise disjoint.)

Since $A_i \in [Y]^{d_i}$ and $A'_i \in [Y]^{d'_i}$, the homogeneity of $[Y]^r$ for $r = 0, 1, \ldots, d$ implies that

$$[Y]^{d_i} \cup [Y]^{d'_i} \subseteq \mathcal{A}_i$$

for all $i = 1, \ldots, h$.

Suppose that $d_j \geqq 1$ and $d_k \geqq 1$ for some $1 \leqq j < k \leqq h$ (or, respectively, that $d'_j \geqq 1$ and $d'_k \geqq 1$ for some $1 \leqq j < k \leqq h$). Let $e = \Sigma_{i=1}^{h} d_i \geqq d$. Choose $T \in [Y]^e$. For each representation $T = \bigcup_{i=1}^{h} T_i$ with $|T_i| = d_i$ for each i, the sets T_i are pairwise disjoint and satisfy $T_i \in [Y]^{d_i} \subseteq \mathscr{A}_i$. The number of such representations is the multinomial coefficient

$$\binom{e}{d_1, \ldots, d_h} \geqq e \geqq d \geqq n.$$

Suppose that $d_j = d$ and $d_i = 0$ for $i \neq j$, and also that $d'_k = d$ and $d'_i = 0$ for $i \neq k$. Since $A_i \neq A'_i$ for some i, it follows that $j \neq k$. By the homogeneity of $[Y]^d$, the fact that $A_j = A'_k = S \in [Y]^d$ implies that

$$[Y]^d \subseteq \mathscr{A}_j \cap \mathscr{A}_k.$$

Choose $T \in [Y]^{2d}$. Every partition of T into two disjoint sets T_j and T_k with $|T_j| = |T_k| = d$ generates a representation $T = \bigcup_{i=1}^h T_i$, where $T_i = \varnothing \in \mathscr{A}_i$ for $i \neq j, k$ and $T_j \in \mathscr{A}_j$ and $T_k \in \mathscr{A}_k$. The number of such representations is the binomial coefficient

$$\binom{2d}{d} \geqq d \geqq n.$$

This proves the Lemma.

Let $\mathscr{A}^* = (\mathscr{A}_1, \ldots, \mathscr{A}_h)$ be an h-tuple of sets $\mathscr{A}_i \subseteq \mathscr{F}(\mathbf{N})$. For $S \in \mathscr{F}(\mathbf{N})$, let $r(S)$ denote the number of representations of S in the form $S = A_1 \cup \cdots \cup A_h$ with $A_i \in \mathscr{A}_i$ for $i = 1, \ldots, h$. Define $\liminf r(S) = s$ if for any sequential ordering of $\mathscr{F}(\mathbf{N})$ in the form $\mathscr{F}(\mathbf{N}) = \{S_n\}_{n=1}^\infty$ we have $\liminf_{n \to \infty} r(S_n) = s$. Define $\limsup r(S)$ similarly.

THEOREM 1. *Let $h \geqq 2$. Let $\mathscr{A}^* = (\mathscr{A}_1, \ldots, \mathscr{A}_h)$ be an asymptotic union system of order h. If* $\liminf r(S) \geqq 2$, *then* $\limsup r(S) = \infty$.

PROOF. Since $r(S) = 0$ or 1 for only finitely many sets $S \in \mathscr{F}(\mathbf{N})$, it follows that the condition of the Lemma holds for all $d \geqq 1$, and Theorem 1 follows immediately.

THEOREM 2. *Let $h \geqq 2$. Let $\mathscr{A}$ be an asymptotic union basis of order h. Then* $\limsup r(S) = \infty$.

PROOF. If $\liminf r(S) \geqq 2$, the result follows from Theorem 1.

Suppose $\liminf r(S) = 1$. Then $r(S) = 1$ for infinitely many sets $S \in \mathscr{A}$. Let $r(S) = 1$ for some $S \neq \varnothing$. If $S = A_1 \cup \cdots \cup A_h$ is the unique representation of S as a union of h elements of $\mathscr{A}$, then $A_1 = \cdots = A_h = S \in \mathscr{A}$. Thus, if $T \subsetneqq S$ then $T \notin \mathscr{A}$. In particular, if $|S| \geqq 2$ and $x \in S$, then $\{x\} \notin \mathscr{A}$. Since $\mathscr{A}$ is an asymptotic basis, it follows that $\{n\} \in \mathscr{A}$ for all but at most finitely many $n \in \mathbf{N}$. Thus, there are at most a finite number of sets S such that $r(S)=1$ and $|S|\geqq 2$. Therefore, the condition of the Lemma holds for all $d\geqq 2$, and Theorem 2 follows immediately.

3. Multiplicative bases and LCM bases

THEOREM 3. *For $h\geqq 2$, let $M(h)$ consist of all pairs (s,t) such that*

$$s=\liminf_{n\to\infty} g_{\mathscr{B}}(n) \quad \textit{and} \quad t=\limsup_{n\to\infty} g_{\mathscr{B}}(n)$$

for some asymptotic multiplicative system $\mathscr{B}=(B_1,\ldots,B_h)$ of order h. Then

$$M(h)=\{(1,t)\mid t\in\mathbf{N}\}\cup\{(s,\infty)\mid s=1,\ldots,h\}.$$

In particular, if $s\geqq 2$, then $t=\infty$.

PROOF. Let $t\in\mathbf{N}\cup\{\infty\}$. Let $\mathscr{B}=(B_1,\ldots,B_h)$, where $B_1=\mathbf{N}$, $B_2=\{2^k \mid 0\leqq k<t\}$, and $B_i=\{1\}$ for $i=3,\ldots,h$. Then $1\leqq g_{\mathscr{B}}(n)\leqq t$ for all $n\in\mathbf{N}$, and $g_{\mathscr{B}}(n)=1$ for n odd, hence $\liminf_{n\to\infty} g_{\mathscr{B}}(n)=1$. If $0\leqq u<t$ and $2^u \parallel n$, then $n=(n\cdot 2^{-k})\cdot 2^k\cdot 1\cdots 1$ for $0\leqq k\leqq u$ and $g_{\mathscr{B}}(n)=u+1$. If t is finite, choose $u=t-1$, and if $t=\infty$, choose u arbitrarily large. In both cases, $\limsup_{n\to\infty} g_{\mathscr{B}}(n)=t$. Therefore, $(1,t)\in M(h)$ for all $t\in\mathbf{N}\cup\{\infty\}$.

For $s\in\{2,\ldots,h\}$, let $\mathscr{B}=(B_1,\ldots,B_h)$ be the system defined by $B_1=\mathbf{N}$, $B_i=\mathbf{P}\cup\{1\}$ for $i=2,\ldots,s$, and $B_i=\{1\}$ for $i=s+1,\ldots,h$. Then $g_{\mathscr{B}}(n)\geqq s$ for all $n\geqq 2$, and $g_{\mathscr{B}}(p)=s$ for all $p\in\mathbf{P}$. If n has k distinct prime factors, then $g_{\mathscr{B}}(n)\geqq k$. It follows that $\limsup_{n\to\infty} g_{\mathscr{B}}(n)=\infty$. Therefore, $(s,\infty)\in M(h)$ for $2\leqq s\leqq h$.

If $\mathscr{B}=(B_1,\ldots,B_h)$ is any multiplicative system, and if $p\in\mathbf{P}$, $p>p_0$, then the only representations $p=b_1b_2\cdots b_h$ are those of the form $b_j=p$ and $b_i=1$ for $i\neq j$. Therefore, $1\leqq g_{\mathscr{B}}(p)\leqq h$ and $s=\liminf_{n\to\infty} g_{\mathscr{B}}(n)\in\{1,\ldots,h\}$.

It only remains to prove that $s\geqq 2$ implies $t=\infty$. Let $\mathscr{B}=(B_1,\ldots,B_h)$ be a multiplicative system with $s\geqq 2$. Let Q be the set of positive, square-free integers. For $q\in Q$, let

$$S(q)=\{p\in\mathbf{P}\mid p\mid q\}\in[\mathbf{P}]^{<\omega}.$$

Define

$$\mathscr{A}_h=\{S(q)\mid q\in Q\cap B_i\}.$$

Then $\mathscr{A}_i\subseteq[\mathbf{P}]^{<\omega}$. Let $\mathscr{A}^*=(\mathscr{A}_1,\ldots,\mathscr{A}_h)$. Applying the Lemma with $X=\mathbf{P}$, we observe that every $S\in[\mathbf{P}]^{<\omega}$ is of the form $S=S(q)$ for some $q\in Q$, and the condition $g_{\mathcal{B}}(n) \geqq 2$ for all $n > n_0$ implies for every $d \geqq 1$ and all but at most finitely many $S \in [\mathbf{P}]^d$ the set theoretic equation $S = A_1 \cup \cdots \cup A_h$ with $A_i \in \mathcal{A}_i$ has at least two solutions. The Lemma implies that for every $n \in \mathbf{N}$ there is a square-free number q such that $g_{\mathcal{B}}(q) \geqq n$, and so $\limsup_{n\to\infty} g_{\mathcal{B}}(n) = \infty$. This proves the Theorem.

The following result was first obtained by Erdős [2] and Nešetřil and Rüdl [6].

THEOREM 4. *Let $h \geqq 2$, and let B be an asymptotic multiplicative basis of order h. Let $g(n)$ denote the number of representations of n in the form $n = b_1 b_2 \cdots b_h$, where $b_i \in B$ for $i = 1, \ldots, h$. Then* $\limsup_{n\to\infty} g(n) = \infty$.

PROOF. If $\liminf_{n\to\infty} g(n) \geqq 2$, the result follows immediately from Theorem 3.

Suppose $\liminf_{n\to\infty} g(n) = 1$. If $g(n) = 1$, then $n = b^h$ for some $b \in B$. If $n > n_0$ and n is square-free, then $g(n) \geqq 2$. It then follows, as in the proof of Theorem 3, that for every k there is a square-free number n such that $g(n) \geqq k$, and so $\limsup_{n\to\infty} g(n) = \infty$. This proves the Theorem.

Analogous results hold for asymptotic LCM systems and bases.

THEOREM 5. *Let $h \geqq 2$, and let $\mathcal{B} = (B_1, \ldots, B_h)$ be an asymptotic LCM system of order h. Let $g'_{\mathcal{B}}(n)$ denote the number of representations of n in the form $n = [b_1, \ldots, b_h]$, where $b_i \in B_i$ for $i = 1, \ldots, h$. Then $\liminf_{n\to\infty} g'_{\mathcal{B}}(n) \geqq 2$ implies that* $\limsup_{n\to\infty} g'_{\mathcal{B}}(n) = \infty$.

PROOF. It suffices to consider only square-free numbers $q \in Q$. To each square-free number q there is associated a finite set $S(q) = \{p \in \mathbf{P} \mid p \mid q\}$ in $[\mathbf{P}]^{<\omega}$ such that if $q = [q_1, \ldots, q_h]$, then $S(q) = S(q_1) \cup \cdots \cup S(q_h)$. Now apply the Lemma exactly as in the proof of Theorem 3. Note that it is important in the case of LCM systems that the Lemma does not assume that the sets $S(q_i)$ are pairwise disjoint.

THEOREM 6. *Let $h \geqq 2$ and let B be an asymptotic LCM basis of order h. Let $g'(n)$ denote the number of representations of n in the form $n = [b_1, \ldots, b_h]$, where $b_i \in B$ for $i = 1, \ldots, h$. Then* $\limsup_{n\to\infty} g'(n) = \infty$.

PROOF. If $\liminf_{n\to\infty} g'(n) \geqq 2$, the result follows from Theorem 5.

Suppose $\liminf_{n\to\infty} g'(n) = 1$. Since B is an asymptotic LCM basis, it follows that $p \in B$ for all but a finite set F of primes p. If $n > 1$ and $g'(n) = 1$, then $n \in B$ and $n = [n, \ldots, n]$ is the unique representation of n as the least common multiple of elements of B. In particular, if p is prime, $p \mid n$, and $p \neq n$, then $p \notin B$.

It follows that if $g'(n) = 1$ and n is not prime, then n is composed only of primes belonging to the finite set F. Therefore, there exists q_0 such that if q is square-free, $q \notin \mathbf{P}$, and $q > q_0$, then $g'(q) \geqq 2$. An application of the Lemma now yields the result.

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